

Unique Representation in Convex Sets by Extraction of Marked Components

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ABSTRACT

After introducing the basic concepts of extraction and marking for convex sets, the following marked representation theorem is established: Let C be a lineally closed convex set without lines, the face lattice of which satisfies some descending chain condition, and let μ be some marking on C . Then every point of C can be represented in unique way as a convex (nonnegative) linear combination of points (directions) of C which are μ -independent, and this representation can be determined by an algorithm of successive extractions. In particular, if C is a finite dimensional closed convex set without lines and μ marks extreme points (directions) only, then the marked representation theorem contains some well-known results of convex analysis as special cases, and it yields in the case where C is a polyhedral triangulation which extends available results on polytopes to the unbounded case. The triangulation of unbounded polyhedra then is applied to a certain class of parametric linear programs.

1. INTRODUCTION

By the theorems of Klee and Carathéodory each point of a d -dimensional closed convex set C containing no line can be represented as convex combination of at most $d+1$ extreme points or directions of C [10,12]. This representation is not unique unless C is a generalized simplex. In this paper we expose a systematic procedure to achieve uniqueness by imposing an additional condition on the representation ("marking") and present a (conceptual) algorithm which determines for every $x \in C$ the uniquely determined "marked" representation.

Our interest in this type of unique representation was motivated by a triangulation problem arising in the economic theory of joint production. To

solve this problem we established an existence and uniqueness theorem for “ordered” extremal representations in convex polytopes [4]. However, without involved technical modifications, the method employed was not applicable to the unbounded case. Since the unbounded case was of special interest in the economic application [5], we developed a different approach, which is presented in this paper. It is equally applicable to bounded and to unbounded closed convex sets.

In order to work out the simple logic of the method, we drop all assumptions which are not inherently required for it. Thus we dispense with finite dimensionality as well as topological assumptions and consider lineally closed convex subsets (of arbitrary real vector spaces) whose face lattice satisfies a descending chain condition.

Following the method of homogenization [12], we first consider the problem for lineally closed pointed convex cones K (Section 2). The basic concepts of *extraction* and *marking* are introduced. Extracting an element $x \in K$ from an element $y \in K$ means subtracting from y the largest real multiple λx such that $y - \lambda x \in K$. A marking μ on K associates with every $y \in K, y \neq 0$, an (extreme) component $\mu(y)$ of y . Given a marking μ on K , the concepts of *marked independence* and *marked basis* are defined. If the face lattice of K satisfies a descending chain condition, the successive extraction of marked components of a given element $x \in K$ yields a representation of x by an independent set of marked points in K , and this representation is unique.

In Section 3 analogous concepts and results are derived for arbitrary lineally closed convex sets C containing no line, in a real vector space E . This is achieved by determining the extraction structure on C which is induced from the extraction structure on the lineal closure of the cone $R_+(\{1\} \times C)$ in $R \times E$. The formulas describing the extraction on C are interpreted geometrically, and their relation to the conic extraction structure is explained. Finally the extraction algorithm is described, and the general existence and uniqueness theorem for marked representations is proved. This theorem enables one to speak of *marked coordinates* for points inside a given convex set.

In Section 4 we illustrate by some applications how the elementary method of extracting marked components can be used as an efficient tool in convex analysis, the theories of convex polyhedra, and linear programming. Firstly we show that the marked representation theorem contains some well-known basic theorems of convex analysis as special cases. Secondly we construct triangulations of a certain type for convex polyhedra, which extend available triangulations for polytopes to the unbounded case. Finally we apply such a triangulation to a class of parametric linear programs and obtain a selection of optimal solutions which depends in a continuous and piecewise affine manner on the parameters of the program.

2. EXTRACTION IN CONVEX CONES

In this section we introduce the concepts of extraction and marking for convex cones and show that under certain conditions every point in the cone admits a unique representation as positive linear combination of marked points. This representation is obtained by successive extraction of marked components.

With a few modifications, our terminology and notation are standard in convex analysis (see [10, 12] for convexity in finite dimensional spaces, [7] for infinite dimensional spaces). By K we denote a convex pointed cone with vertex 0 in some real vector space. We suppose that K is *lineally closed*, or equivalently that the ordering \leq induced by K on $K - K$ is archimedean. The faces of K are those convex subcones F of K which are hereditary, i.e., $0 \leq x \leq y \in F$ implies $x \in F$. Evidently the faces of K are lineally closed, too. The smallest face $F(x)$ containing $x \in K \setminus \{0\}$ is called a *principal face*, and such a face is called an *extreme ray* if $F(x) = R_+ x$. Of course, a face $F \neq \{0\}$ is principal if and only if it is the smallest face $F(x_1, x_2, \dots, x_m)$ containing some finite subset $\{x_1, x_2, \dots, x_m\}$ of K .

Because K is assumed to be pointed and lineally closed, the following definition is meaningful.

DEFINITION 2.1. The functions $\lambda: K \times K \rightarrow [0, \infty]$ and $e: K \times K \rightarrow K$ defined by $\lambda(x, y) = \sup\{\lambda \geq 0; y - \lambda x \in K\}$ and $e(x, y) = y - \lambda(x, y)x$ (setting $\infty \cdot 0 = 0$) are called respectively the *order function* and the *extraction function* of K . The element x is called a *component* of y if $\lambda(x, y) > 0$, and $e(x, y)$ is called the *rest* after extracting x from y .

The following *elementary properties of the order function and the extraction function* are easily derived. For $x, y, z \in K$ and $\alpha > 0$:

$$\lambda(x, y) < \infty \quad \text{for } x \neq 0,$$

$$\lambda(x, y) > 0 \quad \text{if and only if } x \in F(y),$$

$$\lambda(x, \alpha y) = \lambda\left(\frac{1}{\alpha} x, y\right) = \alpha \lambda(x, y), \quad (2.1)$$

$$\lambda(x, y + z) \geq \lambda(x, y) + \lambda(x, z) \quad \text{with equality for } x = y,$$

$$\lambda(x + y, z) \leq \min\{\lambda(x, z), \lambda(y, z)\}$$

and

$$e(x, y) \in F(y) \text{ and } x \notin F(e(x, y)) \quad \text{for } x \neq 0,$$

$$e(x, \alpha y) = \alpha e(x, y) \quad \text{and} \quad e(\alpha x, y) = e(x, y),$$

$$e(x, y + z) \leq e(x, y) + e(x, z) \quad \text{with equality for } x = y, \quad (2.2)$$

$$z + e(x + y, z) \geq e(x, z) + e(y, z).$$

The next definition shows how to iterate the process of extraction by marking for each element of the cone one component which shall be extracted from it.

DEFINITION 2.2. A mapping $\mu: K \rightarrow K$ is called a *marking on K* if for all $x, y \in K$:

- (i) $\mu(x)$ is a component of x and $\mu(x) \neq 0$ for $x \neq 0$;
- (ii) If x is a component of y and $\mu(y)$ is a component of x , then $\mu(x) = \mu(y)$.

If μ is a marking on K , then $e(x) = e(\mu(x), x)$ defines a mapping $e: K \rightarrow K$, called the *marked extraction on K* .

From the definition of a marking μ it follows that $F(x) = F(y)$ implies $\mu(x) = \mu(y)$. Hence μ induces a mapping from the set of principal faces into the cone by

$$\mu(F) := \mu(x) \quad \text{for } F = F(x) \quad (2.3)$$

EXAMPLE 2.3. The convex cone generated by a finite set of positively independent vectors (in some real vector space) is pointed and lineally closed. Choose on each extreme ray just one point $\neq 0$, and let z_1, \dots, z_n be an arbitrary numbering of these points. Assigning to each $x \neq 0$ the point z_i with smallest index i contained in $F(x)$ defines a marking on K .

Consider now the convex cone K together with some marking μ on it. Successive extraction of marked components from an element $x \neq 0$ in K is performed by iterating the marked extraction e as long as the outcome $e^i(x)$ is different from 0. The following lemma exhibits some simple properties of the iterated marked extraction.

LEMMA 2.4. *Let*

$$m(x) = \sup\{i \geq 0; e^i(x) \neq 0\} \text{ for } 0 \neq x \in K$$

and

$$x_i = e^i(x), \quad e_i = \mu(x_i), \quad \lambda_i = \lambda(e_i, x_i) \quad \text{where } 0 \leq i < m(x) + 1. \quad (2.4)$$

Then these sequences have the following properties. For

$$0 \leq i < j < m(x) + 1,$$

$$(i) \quad x_i = \sum_{k=i}^j \lambda_k e_k + x_{j+1} \quad \text{with } \lambda_k > 0, \quad (2.5)$$

$$(ii) \quad F(x_{i+1}) \subsetneq F(x_i) \quad \text{and} \quad e_i \notin F(x_{i+1}). \quad (2.6)$$

In particular, the e_i are all different.

Proof. (i): By definition $x_{i+1} = e(x_i) = e(\mu(x_i), x_i) = e(e_i, x_i) = x_i - \lambda_i e_i$. Hence $x_i - x_{j+1} = \sum_{k=i}^j (x_k - x_{k+1}) = \sum_{k=i}^j \lambda_k e_k$. Furthermore $\lambda_k > 0$, because $\mu(x_k)$ is a component of x_k .

(ii): By (2.2) it follows that $e_i \notin F(e(e_i, x_i)) = F(e(x_i)) = F(x_{i+1})$. By (i), $x_{i+1} \leq x_i$ and therefore $F(x_{i+1}) \subset F(x_i)$. Because e_i is a component of x_i , it follows from (2.1) that $e_i \in F(x_i)$. Hence $F(x_{i+1}) \subsetneq F(x_i)$.

Finally, $e_j = \mu(x_j) \in F(x_j) \subset F(x_{i+1})$ if $0 \leq i < j < m(x) + 1$. Since $e_i \notin F(x_{i+1})$, we have $e_j \neq e_i$. ■

To assure that the marked extraction ends for each point of K after finitely many steps, the following finiteness condition is appropriate.

DEFINITION 2.5. The convex cone K is said to be of *finite type* if there exists no infinite properly descending chain of principal faces $F_1 \supsetneq F_2 \supsetneq \cdots$ of K .

If K is finite dimensional, then $\dim K$ is an upper bound for the lengths of all properly descending chains of faces $\neq \{0\}$. However, the existence of such an upper bound does not imply that K is finite dimensional. Furthermore there exist cones K of finite type for which the lengths of properly descending

chains of principal faces is not bounded above. Such cones may even possess *infinite* properly descending chains of *arbitrary* faces.

If K is of finite type, then the set of e_i 's produced by marked extraction according to (2.4) is a μ -independent set in the sense of the following definition.

DEFINITION 2.6. A nonempty and finite subset M of the convex cone K is called μ -independent with respect to a marking μ on K , if there exists some numbering $M = \langle a_0, a_1, \dots, a_m \rangle$ and a properly descending chain of principal faces $F_0 \supsetneq F_1 \supsetneq \dots \supsetneq F_m$, such that $a_i = \mu(F_i)$ for all i . A maximal μ -independent subset is called a μ -base.

In other words, a μ -independent set is the set of marked points taken from a descending chain of principal faces. If we want to indicate the numbering mentioned in the above definition, we shall write $M = (a_0, a_1, \dots, a_m)$. It follows immediately that $M = (a_0, \dots, a_m)$ is μ -independent if and only if

$$\begin{aligned} a_i &\notin F(a_{i+1}, \dots, a_m) & \text{for } 0 \leq i \leq m-1, \\ a_i &= \mu(F(a_i, \dots, a_m)) & \text{for } 0 \leq i \leq m. \end{aligned} \tag{2.7}$$

The following lemma shows that a μ -independent set can be reconstructed from any positive combination of its elements by marked extraction.

LEMMA 2.7. Let $M = (a_0, \dots, a_m)$ be a μ -independent set with respect to a marking μ on K . The marked extraction procedure (2.4) when applied to $x = \sum_{i=0}^m \alpha_i a_i$ with all $\alpha_i > 0$ yields $m(x) = m$ and $x_i = \sum_{j=i}^m \alpha_j a_j$, $e_i = a_i$, $\lambda_i = \alpha_i$ for $0 \leq i \leq m$.

Proof. By induction with respect to i we prove $e_i = a_i$, $\lambda_i = \alpha_i$, $x_{i+1} = \sum_{j=i+1}^m \alpha_j a_j$ for $0 \leq i \leq m$ (setting $\sum_{\emptyset} = 0$). If $i = 0$, then $F(x_0) = F(a_0, \dots, a_m)$ because $x_0 = x$. Hence, using (2.4) we get $e_0 = \mu(x_0) = \mu(F(x_0)) = \mu(F(a_0, \dots, a_m)) = a_0$ by (2.7). From (2.1) then it follows that

$$\frac{1}{\alpha_0} \lambda(e_0, x_0) = \lambda(\alpha_0 e_0, x_0) = 1 + \lambda\left(\alpha_0 a_0, \sum_{j=1}^m \alpha_j a_j\right),$$

and therefore $\lambda_0 = \lambda(e_0, x_0) = \alpha_0$ because $a_0 \notin F(a_1, \dots, a_m)$. Furthermore, $x_1 = e(x_0) = e(\mu(x_0), x_0) = x_0 - \alpha_0 a_0 = \sum_{j=1}^m \alpha_j a_j$

Assume now that the assertion holds for $i \leq m-1$. Thus $F(x_{i+1}) = F(a_{i+1}, \dots, a_m)$, and therefore $e_{i+1} = a_{i+1}$ by (2.4) and (2.7). Applying (2.1), we get

$$\begin{aligned} \frac{1}{\alpha_{i+1}} \lambda(e_{i+1}, x_{i+1}) &= \lambda(\alpha_{i+1} e_{i+1}, x_{i+1}) \\ &= 1 + \lambda\left(\alpha_{i+1} e_{i+1}, \sum_{j=i+2}^m \alpha_j a_j\right), \end{aligned}$$

and therefore $\lambda_{i+1} = \lambda(e_{i+1}, x_{i+1}) = \alpha_{i+1}$. Finally,

$$\begin{aligned} x_{i+2} &= e(x_{i+1}) = e(\mu(x_{i+1}), x_{i+1}) \\ &= x_{i+1} - \alpha_{i+1} e_{i+1} = \sum_{j=i+2}^m \alpha_j a_j. \end{aligned}$$

In particular we conclude for $i = m-1$ and $i = m$ that $x_m = \alpha_m a_m \neq 0$ and $x_{m+1} = 0$. Hence $m(x) = m$. \blacksquare

Combining the two lemmas, we arrive at the following representation theorem for convex cones.

THEOREM 2.8. *Let K be a convex cone which is pointed, lineally closed, and of finite type, and let μ be some marking on K . Then every point x in K , $x \neq 0$, has exactly one marked representation, that is, a representation*

$$x = \sum_{a \in M} \lambda_a a \quad \text{with } M \text{ } \mu\text{-independent and } \lambda_a > 0 \text{ for } a \in M \quad (2.8)$$

This representation is given by marked extraction, that is, $M = (e_0, e_1, \dots, e_{m(x)})$ and $\lambda_{e_i} = \lambda(e_i, e^i(x))$ with $e_i = \mu(e^i(x))$. Furthermore, the length $|M|$ of the representation does not exceed the maximal length of properly descending chains of principal faces in the subcone $F(x)$. In particular $|M| \leq \dim F(x)$.

If K is any (not necessarily pointed) closed convex cone in R^n , then K may be written as a direct sum of its lineality space $L = K \cap (-K)$ and a convex cone K' which is closed, finite dimensional and pointed [12]. Hence, the above theorem can be combined with the unique representation by some "marked" vector space basis, to get a unique representation theorem for finite dimensional convex cones which are not pointed.

3. MARKED REPRESENTATIONS IN CONVEX SETS AND THE EXTRACTION ALGORITHM

In this section we deduce an existence and uniqueness theorem for marked representations in lineally closed convex sets from the corresponding theorem for cones (cf. Section 2) and show that the representations can be determined by an algorithm of successive extractions.

Let C be a nonempty *lineally closed convex set without line* in a real vector space E . Whenever there is no risk of ambiguity we identify C with $C_1 := \{1\} \times C \subset R \times E$. By $D = D(C)$ we denote the (lineally closed, convex, pointed) recession cone of C_1 . In particular $D \subset \{0\} \times E$, and hence $C \cap D = \emptyset$. A typical element of $R \times E$ is written as $x = (x_0, \mathbf{x})$. By \bar{M} we denote the lineal closure of $M \subset R \times E$ and by $K(N)$ the cone $R_+((1) \times N) \subset R \times E$ for any given subset $N \subset C$.

The following lemma relates the faces of C to those of $\overline{K(C)}$.

LEMMA 3.1. *For every face F of C ,*

$$\overline{K(F)} = K(F) \cup D(F) \quad (3.1)$$

is a face of $K := \overline{K(C)}$. Each face of K is either of this form or a face of D .

Proof. (3.1) is an easy generalization of a well-known equality for closed convex sets in R^n (see [10, p. 63]). Furthermore it is straightforward to prove that for $x, y \in K$, $x + y = z \in \overline{K(F)}$, F any face of C , we have

- (1) $x, y \in D(F)$ if $z_0 = 0$;
- (2) $x \in F, y \in D(F)$ if $z_0 > 0$ and $y_0 = 0$;
- (3) $x, y \in K(F)$ if $x_0 > 0$ and $y_0 > 0$.

Hence $\overline{K(F)}$ is a face of K if F is a face of C . Conversely let S be a face of K which is not contained in D . Then $S \cap C_1 \neq \emptyset$ is of the form $\{1\} \times F$ where F is a face of C . Clearly, $\overline{K(F)} \subset S$ and $S \cap K(C) \subset \overline{K(F)}$. It only remains to verify $S \cap D \subset D(F)$. But this is immediate because $\{1\} \times (F + y) \subset S \cap C_1 = \{1\} \times F$ for all $(0, y) \in S$. ■

Let \mathcal{F}_C (\mathcal{F}_D) be the face lattice of C (D). Lemma 3.1 shows that the face lattice \mathcal{F}_K of K induces the following order relation on $\mathcal{F} := \mathcal{F}_C \cup \mathcal{F}_D$:

$$F < F' \quad : \Leftrightarrow \quad \begin{cases} F \subset F' & \text{if } F, F' \in \mathcal{F}_C \text{ or } F, F' \in \mathcal{F}_D, \\ F \subset D(F') & \text{if } F \in \mathcal{F}_D \text{ and } F' \in \mathcal{F}_C. \end{cases} \quad (3.2)$$

If M is a subset of $C \cup D$, $M \cap C \neq \emptyset$, then we denote by $F(M)$ the smallest $F \in \mathcal{F}$ such that

$$M \cap C \subset F \quad \text{and} \quad M \cap D \subset D(F) \quad (3.3)$$

[if $M \subset D$, $F(M)$ has been defined in Section 2]. ■

The following corollary implies that (\mathcal{F}, \prec) is a lattice ("extended face lattice of C'' [1]).

COROLLARY 3.2.

(i) $k: \mathcal{F} \rightarrow \mathcal{F}_K$, defined by

$$k(F) = \begin{cases} \overline{K(F)} & \text{if } F \in \mathcal{F}_C, \\ F & \text{if } F \in \mathcal{F}_D, \end{cases} \quad (3.4)$$

is a lattice isomorphism of (\mathcal{F}, \prec) onto (\mathcal{F}_K, \subset) .

(ii) If $M \subset C \cup D$, then $k(F(M))$ is the smallest face of K which contains M .

The preceding considerations indicate that we have to take into account both the point set C and the cone D of direction vectors if we want to develop the concepts of Section 2 for lineally closed convex sets. We first proceed to derive an extraction structure on C from the extraction e_K on the cone K .

Let $\pi: K \rightarrow C \cup D$ be defined by

$$\pi(x) = \begin{cases} x_0^{-1}x & \text{if } x_0 > 0 \\ x & \text{if } x_0 = 0 \end{cases} \quad [x = (x_0, x) \in K],$$

and $\lambda(x, y)$ for $x, y \in K$ be as in Definition 2.1. Then a straightforward but tedious calculation yields the following

LEMMA 3.3. For $x, y \in C_1 \cup D$, $x \neq 0$, the following equations hold:

(i)

$$\lambda(x, y) = \begin{cases} \sup\{\gamma \in [0, 1]; \exists z \in C_1: y = \gamma x + (1 - \gamma)z\} & \text{if } x, y \in C_1, \\ \max\{\gamma \in [0, \infty); y - \gamma x \in C_1\} & \text{if } x \in D, y \in C_1, \\ 0 & \text{if } x \in C_1, y \in D, \\ \max\{\gamma \in [0, \infty); y - \gamma x \in D\} & \text{if } x, y \in D, \end{cases} \quad (3.5)$$

(ii)

$$\pi e_K(x, y) = \begin{cases} [1 - \lambda(x, y)]^{-1} [y - \lambda(x, y)x] & \text{if } x, y \in C_1, \quad y - x \notin D, \\ y - \lambda(x, y)x & \text{otherwise.} \end{cases} \quad (3.6)$$

DEFINITION 3.4. The mapping $e(\cdot, \cdot) = \pi e_K(\cdot, \cdot)$ restricted to $(C_1 \cup D) \times (C_1 \cup D)$ is called the (extended) *extraction function* on C . $\lambda(\cdot, \cdot)$ restricted to $(C_1 \cup D) \times (C_1 \cup D)$ is called the (extended) *order function* on C . An element $x \in C_1 \cup D$ is called a *component* of $y \in C_1 \cup D$ if $\lambda(x, y) > 0$.

The preceding lemma presents explicit formulas for λ and e in terms of C itself. In particular it implies for $x, y \in C$ that

$$\lambda(x, y) = 1 \iff y - x \in D(C), \quad (3.7)$$

and for $x, y \in C \cup D$ that

$$x \text{ is a component of } y \iff x \in F(y) \cup DF(y) \quad [\text{or } x \in F(y) \text{ if } y \in D].$$

The diagrams in Figure 1 illustrate the geometrical meaning of the extraction function on convex sets for three different cases. In all these cases we embed $C \subset R^2$ in R^3 in order to picture its recession cone simultaneously.

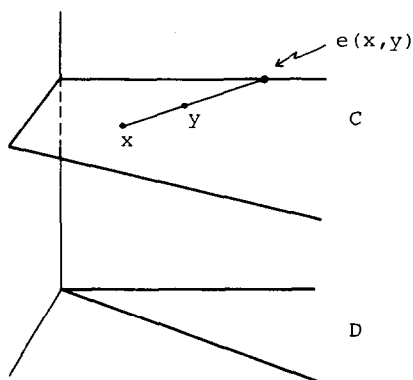
If C is *linearly bounded* only the first case can occur. Then $\lambda(x, y)$ is the largest $\gamma \in [0, 1]$ such that

$$\exists z \in C: y = \gamma x + (1 - \gamma)z \quad (3.8)$$

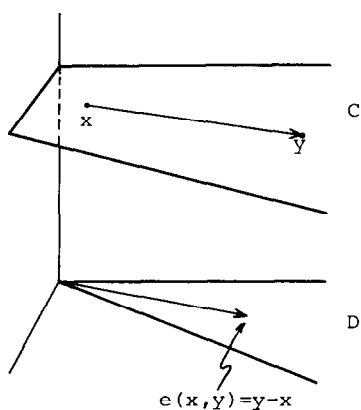
and $e(x, y)$ is the $z \in C$ satisfying (3.8) for $\gamma = \lambda(x, y)$. This case represents the pure version of convex set extraction. For *unbounded* convex sets e can be viewed as a combination of the cone extraction introduced in Section 2 and the convex set extraction just considered. Both types of extraction have to be carefully distinguished, as the following remark illustrates.

REMARK 3.5. If C is a cone with vertex 0, then the convex set extraction e on C of Definition 3.4 does *not* coincide with the cone extraction e_C on C (def. 2.1); see Figure 2.

$$x, y \in C, e(x, y) \in C:$$



$$x, y \in C, e(x, y) \in D:$$



$$y \in C, x \in D, e(x, y) \in C:$$

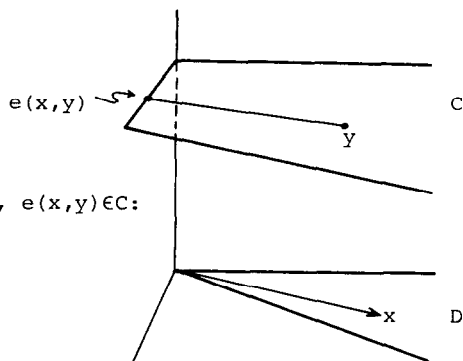


FIG. 1.

The relation between the two types of extraction is clarified by the following formula:

$$(1, e_C(x, y)) = e((0, x), (1, y)),$$

where we distinguish between $x \in C$ and $(1, x) \in C_1$. Hence the cone extraction e_C can be considered as restriction of the convex set extraction e to $D \times C$: in $e_C(x, y)$ the first argument plays the role of a *direction vector* and the second one the role of a *point* in C . Geometrically the dichotomy of direction vectors and points of C can be removed by identifying the direction vectors with corresponding points at infinity.

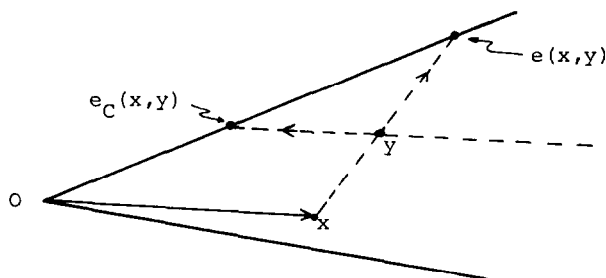


FIG. 2.

Let us now proceed to describe the extraction algorithm on the convex set C . We call C of *finite type* if every descending chain of principal faces in $(\mathcal{F}, <)$ becomes stationary. Because of Corollary 3.2, C is of finite type if and only if K is of finite type.

A map $\mu: C \cup D \rightarrow C \cup D$ is called a *marking* on C if it satisfies for all $x, y \in C \cup D$ the following conditions:

- (i) $\mu(x)$ is a component of x and $\mu(x) \neq 0$ for $x \neq 0$;
- (ii) if x is a component of y and $\mu(y)$ is a component of x , then $\mu(x) = \mu(y)$.

In particular, a marking satisfies

$$F(x) = F(y) \Rightarrow \mu(x) = \mu(y);$$

hence μ induces a mapping from the set of extended principal faces into $C \cup D$ by

$$\mu(F) := \mu(x) \quad \text{if } F = F(x).$$

EXAMPLE 3.6. Suppose that C is a convex polyhedron generated by a finite sequence v_1, \dots, v_r of points or direction vectors. Then we can introduce a marking μ on C —as in Example 2.3—by defining $\mu(x) := v_{i_x}$, where

$$i_x := \begin{cases} \min\{i; v_i \in F(x) \cup D(F(x))\} & \text{if } x \in C, \\ \min\{i; v_i \in F(x)\} & \text{if } x \in D \setminus \{0\} \end{cases}$$

for $x \neq 0$ and $\mu(0) := 0$.

Given any marking μ on a lineally closed convex set C without lines, the associated marked extraction

$$e(\cdot): C \cup D \rightarrow C \cup D$$

is defined by

$$e(x) = e(\mu(x), x) \quad \text{for } x \in C \cup D.$$

For a given initial point $x \in C$ the extraction algorithm on C generates the sequences $x_k = e^k(x)$, $k \geq 0$ and $e_k = \mu(x_k)$, $k \geq 0$. The algorithm proceeds by the following steps:

$$1. \quad \text{Start} \quad k = 0, \quad \beta_k = 1, \quad x_k = x$$

$$2. \quad e_k = \mu(x_k)$$

$$3. \quad \gamma_k = \lambda(e_k, x_k)$$

$$4. \quad \lambda_k = \beta_k \gamma_k$$

$$5. \quad \text{If } e_k \in C \text{ and } x_k \in C \text{ and } \gamma_k < 1$$

$$x_{k+1} = (1 - \gamma_k)^{-1}(x_k - \gamma_k e_k)$$

$$\beta_{k+1} = (1 - \gamma_k) \beta_k$$

$$6. \quad x_{k+1} = x_k - \gamma_k e_k \quad (3.9)$$

$$\beta_{k+1} = \beta_k$$

$$7. \quad \text{If } x_{k+1} = 0$$

$$m(x) = k$$

Go to 10

$$8. \quad k = k + 1$$

$$9. \quad \text{Go to 2}$$

$$10. \quad \text{Write } x, \quad m(x), \quad e_i, \lambda_i \text{ for } i = 0, \dots, m(x).$$

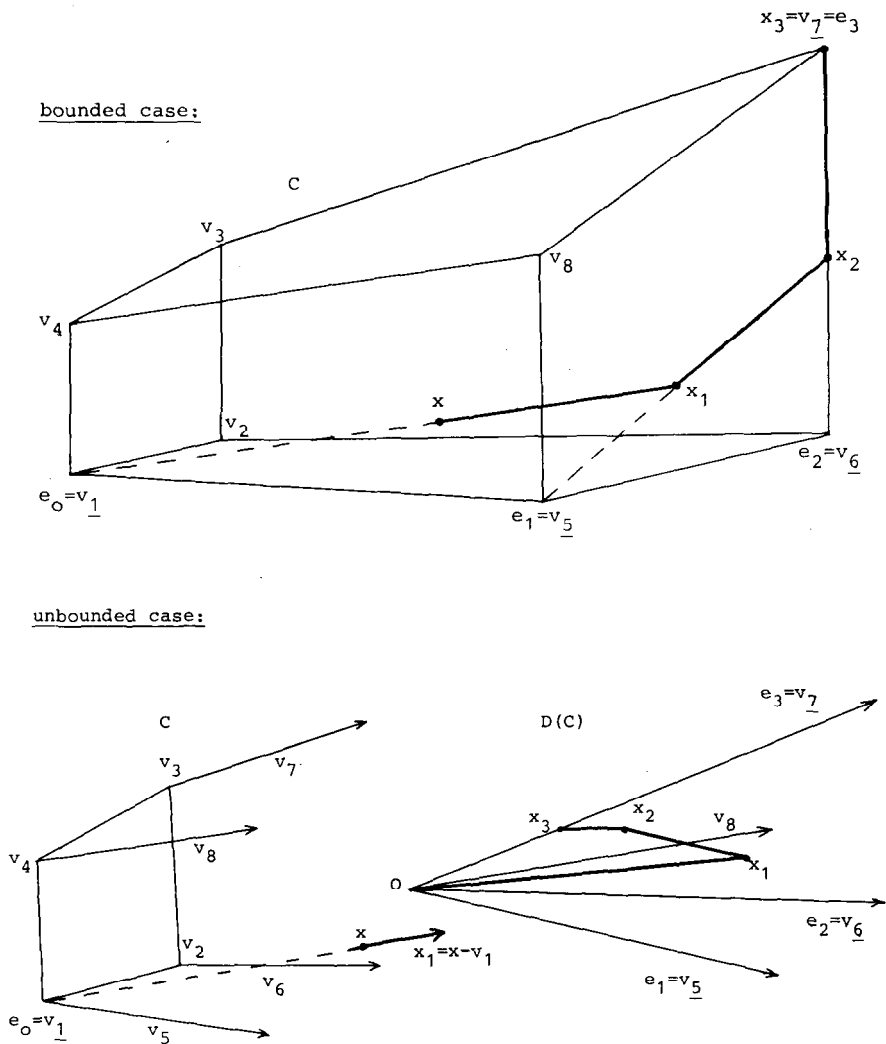


FIG. 3.

We shall see that the extraction algorithm ends after finitely many steps if C is of finite type.

The working of the extraction algorithm is illustrated by the diagrams in Figure 3.

A finite nonempty subset $M \subset C \cup D$ is called μ -independent if there exists an ordering $M = \{a_0, a_1, a_2, \dots, a_m\}$ and a properly descending chain of

principal extended faces $F_0 > F_1 > \cdots > F_m$ such that

$$a_i = \mu(F_i), \quad i = 0, \dots, m.$$

A maximal μ -independent subset $M \subset C \cup D$ is called a μ -base of C . The following representation theorem is the counterpart of Theorem 2.8 for convex sets.

THEOREM 3.7 (Existence and uniqueness of marked representations).

(a) Let C be a lineally closed convex set of finite type which does not contain a line and μ a marking on C . Then every point $x \in C$ possesses a uniquely determined marked representation with respect to μ , i.e. a representation of the form

$$x = \sum_{a \in M} \lambda_a a, \quad \lambda_a > 0 \text{ for } a \in M, \quad \sum_{a \in M \cap C} \lambda_a = 1, \quad (3.10)$$

where M is μ -independent.

(b) This representation is determined by the extraction algorithm (3.9).

(c) The length $|M|$ of the representation does not exceed the maximal length of properly descending chains of principal extended faces of $F(x)$. In particular $|M| \leq \dim F(x) + 1$.

Proof. We identify C and C_1 . Then μ induces a marking $\bar{\mu}$ on the cone K by the definition

$$\bar{\mu}(x) = \mu(\pi(x)) \quad \text{for } x \in K \setminus \{0\}. \quad (3.11)$$

Denote by $e_K(\cdot)$ [$e(\cdot)$] the marked extraction on K [C] associated with $\bar{\mu}$ [μ]. By Definition 3.4 we have $\pi e_K(x) = e(x)$ for $x \in C \cup D$. Since K is of finite type, Theorem 2.8 is applicable. Hence the marked extraction for cones applied to $x \in C$ generates a $\bar{\mu}$ -independent set $M = \{a_0, \dots, a_m\}$ with

$$a_i = \bar{\mu}(z_i), \quad z_i = e_K^i(x) \quad \text{for } i = 0, \dots, m = m(x) \quad (3.12)$$

such that

$$x = \sum_{i=0}^m \lambda_{a_i} a_i, \quad \lambda_{a_i} > 0 \text{ for } i = 0, \dots, m. \quad (3.13)$$

Because of (3.11) and (3.12) we have $M \subset C \cup D$. By Corollary 3.2, M is μ -independent. Finally $x \in C$ implies that $\sum_{a_i \in M \cap C} \lambda_{a_i} = 1$. This proves the existence statement in (a). The uniqueness of the representation (3.10) follows directly from the uniqueness statement of Theorem 2.8, since every μ -independent set $M \subset C \cup D$ is $\bar{\mu}$ -independent too. This completes the proof of (a).

Assertion (c) is a direct consequence of the corresponding statement in Theorem 2.8 because of Corollary 3.2.

It remains to prove (b). Firstly, observe that

$$\pi z_i = \pi e_K^i(x) = (\pi e_K)^i(x) = e^i(x) \quad \text{for } i = 0, \dots, m$$

because e_K is positively homogeneous. Therefore (3.12) implies $e_i = \mu(e^i(x)) = a_i$ for $i = 0, \dots, m$, i.e., the extraction algorithm for convex sets applied to x generates the same μ -independent set M . It remains to prove that the numbers λ_k computed by the algorithm coincide with λ_{a_k} . For this it suffices to show that for $k = 0, \dots, m$

$$x = \sum_{i=0}^k \lambda_i e_i + \beta_{k+1} x_{k+1} \quad (3.14)$$

For $k = 0$ this is valid because $x = \gamma_0 e_0 + (1 - \gamma_0) x_1$, $\beta_0 = 1$, $\beta_1 = 1 - \gamma_0$, $\lambda_0 = \gamma_0$. Suppose that (3.14) has been proved for $k - 1$. Then

$$x = \sum_{i=0}^{k-1} \lambda_i e_i + \beta_k [\gamma_k e_k + (1 - \gamma_k) x_{k+1}]$$

if step 5 is applied, and

$$x = \sum_{i=0}^{k-1} \lambda_i e_i + \beta_k [\gamma_k e_k + x_{k+1}]$$

if step 6 is applied in the k th run of the algorithm. Since $\lambda_k = \beta_k \gamma_k$, (3.14) follows by induction. ■

The theorem on the existence and uniqueness of marked representations enables one to define local coordinates on a convex set in the following sense.

DEFINITION 3.8. Under the assumptions of Theorem 3.7, let A_μ denote the set of values $\{\mu(x); x \in C \cup D(C) \setminus \{0\}\}$ of the marking μ , and let $M(x)$

denote the μ -independent set associated with $x \in C$. Then $k(x) \in R_+^{\Lambda_\mu}$ defined by

$$k_a(x) = \begin{cases} \lambda_a, & a \in M(x) \\ 0, & a \notin M(x) \end{cases} \quad \text{for } a \in A_\mu$$

is called the vector of *marked coordinates* of x with respect to μ . The mapping $k: C \rightarrow R_+^{\Lambda_\mu}$ is called the *coordinate function* on C induced by μ .

Using this definition, the representation (3.10) may be also restated as

$$x = \sum_{a \in A_\mu} k_a(x) a \quad \text{for all } x \text{ in } C. \quad (3.15)$$

From Theorem 3.7 one obtains easily the following properties of the coordinate function.

COROLLARY 3.9. *Under the assumptions of Theorem 3.7 the following properties hold:*

(i) *The coordinate function is affine on convex subsets spanned by marked bases, i.e. on sets of the form $\text{conv}(M \cap C) + \text{cone}(M \cap D(C))$ where M is a marked basis with respect to μ .*

(ii) *If A_μ is finite, then the coordinate function is continuous (with respect to the usual vector space topology on the finite dimensional linear subspace generated by C).*

4. APPLICATIONS

In this section we illustrate by some applications how the elementary method of extracting marked components can be used as an efficient tool in convex analysis and the theory of convex polyhedra. Firstly we show that the existence statement of the marked representation Theorem 3.7 when specialized to the case of finite dimensions contains three well-known and basic representation theorems of convex analysis. Moreover, the iterative extraction of components provides a conceptual algorithm for constructing such representations.

On the other side, the uniqueness statement of the marked representation theorem when specialized to convex polyhedra P with an extremal marking (Lemma 4.1) yields a subdivision of P into generalized simplices which

extends available triangulations of polytopes to the unbounded case. Finally, this triangulation is applied to a class of parametric linear programs. It defines a simplicial subdivision of the parameter space into cones and yields a selection of basic optimal solutions which is continuous and positively homogeneous on the parameter space and additive on every cone of the subdivision. Furthermore, these optimal solutions are given by marked coordinates and can therefore be determined by the extraction algorithm.

For any application of the representation Theorem 3.7 we need a marking on the convex set C . As in convex analysis the representation by extreme elements plays a major role, we are especially interested in markings which are extremal in the following sense. Denote by $E(C)$ the set of extreme points and extreme directions of C . We call a marking μ on C an *extremal marking* with reference point $c \in C \cup D(C) \setminus \{0\}$ if

$$A_\mu \subset E(C) \cup \{c\} \quad \text{and} \quad \mu(C) = c.$$

LEMMA 4.1. *Every lineally closed convex set $C \neq \emptyset$ of finite type which contains no lines has at least one extreme point and, if $D(C) \neq \{0\}$, at least one extreme direction. Moreover for any $c \in C \cup [D(C) \setminus \{0\}]$ there exists an extremal marking with reference point c .*

Proof.

(i) In a first step we show by extraction that each $x \in C$, and each $x \in D(C)$ with $x \neq 0$, has a component in $E = E(C)$. Let $x \in C$, and suppose x is not an extreme point. Because C contains no line, there exists a $y \in F(x)$ such that $x - y \notin D(C)$. By extracting y from x we get a point $x_1 = e(y, x)$ in C such that $y \notin F(x_1)$. If x_1 is an extreme point, we are ready; otherwise we proceed with x_1 as with x before. We continue this process and obtain a sequence x, x_1, x_2, \dots in C such that $F(x) \supsetneq F(x_1) \supsetneq F(x_2) \supsetneq \dots$. After a finite number of steps this process breaks off, because C is assumed to be of finite type. If x_n is the last term, then x_n is an extreme point and a component of x . Analogous reasoning can be applied to $x \in D(C)$, $x \neq 0$.

(ii) Let E^* be the set obtained from $E \cup \{c\}$ by normalizing, i.e. by selecting from each extreme ray of $D(C)$ exactly one point different from 0. By the well-ordering principle there exists a well-ordering of E^* with c as its least element. For $x \in C \cup D(C) \setminus \{0\}$ define $\mu(x)$ to be the least element of $F(x) \cap E^*$, the latter set being nonempty because of (i). It is easily verified that μ is an extremal marking on C with reference point c . ■

Now let $C \neq \emptyset$ be a finite dimensional closed convex set containing no lines. Choose an arbitrary extremal marking on C with reference point in $E(C)$. Then the existence statement of Theorem 3.7 immediately yields the following well-known representation theorems of convex analysis (see e.g. [10, pp. 155, 166, 171] and [12, pp. 35, 46, 114]). Every point of C is a convex combination of elements in $E(C)$ (theorem of Klee). If, for a set S of points and directions, $C = \text{conv } S$, then $E(C) \subset S$, and hence by statement (c) of Theorem 3.7 every point of C is a convex combination of at most $\dim C + 1$ elements in S (theorem of Carathéodory). Assume C is a polyhedron, i.e., C is the intersection of finitely many closed half spaces. Because each face of C and $D(C)$ is given by the intersection of finitely many hyperplanes, there are only finitely many extreme points and extreme directions, and hence C is the convex hull of finitely many points and directions (finite basis theorem).

Thus we see that the method of successive extractions yields a simple and constructive proof for each of these three basic theorems. It should be noted that a marking on C [or a well-ordering of $E(C)$] is not needed for this purpose. Indeed, it suffices to extract successively *arbitrary* extreme components [cf. proof of Lemma 4.1, part (i)] in order to obtain the corresponding representations.

The marking becomes important when the uniqueness of the representation plays a role. In the following we derive a triangulation of (convex) polyhedra from the uniqueness statement of Theorem 3.7.

LEMMA 4.2. *Let C be a polyhedron containing no lines, and let μ be some marking on C . Then a μ -independent subset M of C is a μ -base if and only if $\dim M = \dim C$.*

Proof.

(i) If M is μ -independent, then M is affinely independent (because of the uniqueness statement in Theorem 3.7) and therefore

$$|M| - 1 = \dim M \leq \dim C.$$

In particular, if $\dim M = \dim C$ then M is necessarily maximal.

(ii) We show that for $\dim M < \dim C$ there exists a μ -independent subset N strictly containing the given μ -independent subset M . If $\dim M < \dim C$, then there exists some $x \in C$ such that $x \notin \text{aff } M$. Define

$$y = \sum_{a \in M} \frac{1}{|M \cap C|} a.$$

The sequence (y_n) in $C \subset R^d$ defined by

$$y_n = \frac{1}{n}x + \left(1 - \frac{1}{n}\right)y \quad \text{for } n \geq 1$$

converges to y . Since C is assumed to be a polyhedron, it has finitely many faces, and therefore A_μ must be finite. From the marked representation Theorem 3.7 we conclude that infinitely many points of $\{y_n\}$ have a representation by one and the same μ -independent subset N . Hence the limit point y has a representation by a subset of N , which by Theorem 3.7 must coincide with M . Thus $M \subset N$, and $M \neq N$ because $x \notin \text{aff } M$. ■

The marked representation theorem together with Lemmata 4.1 and 4.2 yields immediately the following triangulation for (possibly unbounded) polyhedra. (For the case of a polytope, i.e. a bounded polyhedron, see [4], [2], [6], [11]. For the notion of a generalized, i.e. unbounded, simplex see [10, p. 154].)

THEOREM 4.3. *Let C be a polyhedron containing no lines, and let μ be an extremal marking on C with reference point c . The finite collection Σ of all convex subsets of C spanned by the μ -bases is an extremal triangulation with reference point c in the following sense:*

- (i) Σ consists of generalized simplices having the same dimension as C .
- (ii) C is the union of the simplices in Σ .
- (iii) The intersection of two simplices in Σ is a face of both of them.
- (iv) Every simplex S in Σ satisfies $c \in E(S) \subset E(C) \cup [c]$, where $[c] = \langle c \rangle$ ($[c] = R_+c$) if $c \in C$ ($c \in D(C)$).

REMARK 4.4. The general case of a polyhedron P with lineality space L can be treated in the following way. Let $c \in P$, B be an arbitrarily chosen vector space basis of L , and L' be any algebraic complement of L in the linear span of P . Then $C = P \cap (L' + c)$ is a polyhedron containing no lines. Let Σ be a triangulation of C with reference point c according to the above theorem, and denote by Σ_L the collection of all simplicial cones K in L generated by subsets of $B \cup -B$ consisting of $|B|$ elements. Then the collection $\Sigma_p = \{K + S; K \in \Sigma_L, S \in \Sigma\}$ satisfies conditions (i)–(iii) of the above theorem with respect to P and satisfies condition (iv) in the form $c \in E(S_p) \subset E(C) \cup [c] \cup R \cdot B$ for every $S_p \in \Sigma_p$.

The theorems of Klee and Carathéodory together, when applied to a polyhedron C containing no lines, amount to saying that every point of C is

contained in some full dimensional generalized simplex sharing its extreme points and extreme directions with the polyhedron. The above Theorem 4.3 (keeping c an extreme point of C) shows moreover that simplices of this type can be selected to form a simplicial subdivision of the polyhedron.

Finally, we shall apply the triangulation Theorem 4.3 to parametric linear programming. Consider the linear program

$$f(x) = \inf\{c\lambda; A\lambda = x, \lambda \geq 0\}, \quad (P)$$

where on the right hand side x varies in the convex cone $X = \{A\lambda; \lambda \geq 0\}$ of feasible vectors, and $A \in R^{m \times n}$, $c \in R^n$ are fixed, A being of rank m . We shall impose the following conditions on A and c :

(N) $A\lambda = 0$ for $\lambda \geq 0$ implies $c\lambda \geq 0$, and $c\lambda = 0$ only if the i th column A_i of A and the i th component c_i of c are both 0 for $\lambda_i \neq 0$.

This condition means that f is a real valued function the epigraph of which contains no lines [see step (1) in the proof of the following theorem]. For that case we shall study how solutions to the program (P) may depend on the parameter x by triangulating the (unbounded) epigraph of f and employing the corresponding coordinate function.

THEOREM 4.5. *Let (P) be a parametric linear program as specified above which satisfies condition (N). Then there exist a simplicial subdivision Σ of X and a function λ on X with values in R^I , I being a subset of $\{1, \dots, n\}$, having the following properties:*

(i) Σ consists of pointed convex cones with vertex 0 generated by m linearly independent columns from $\{A_i, i \in I\}$.

(ii) λ solves the program (P), that is, for every $x \in X$ we have $f(x) = c\lambda(x)$ and $x = A\lambda(x)$, $\lambda(x) \geq 0$ (where only the components of c and the columns of A with an index in I are taken into account).

(iii) λ is continuous and positively homogeneous on X , additive on every cone of Σ , and $\lambda(A_i)$ is the i th unit vector in R^I for $i \in I$. (In particular, $\lambda(x)$ is a basic optimal solution of (P) for every parameter value x .)

Proof.

(1) Let us first see that $C = \{(x, r); x \in X, r \in R, f(x) \leq r\}$, the epigraph of f , is a polyhedral convex cone containing no lines. The function f is subadditive and positively homogeneous on X and, since $f(0) = 0$ by (N), real valued. Thus f is a real valued finitely generated function the epigraph of which is a polyhedral cone in R^{m+1} generated by $(0, 1)$ and (A_i, c_i) , $i = 1, \dots, n$

(cf. [10, p. 173]). If $(x, r) \in C \cap (-C)$, then $x = A\lambda = -A\mu$, $r = c\lambda + s = -c\mu - t$ with $\lambda, \mu \in R_+^n$, $s, t \in R_+$. From condition (N), it follows that $x = 0$, $r = 0$ and hence C contains no lines. [Conversely, if f is real valued and $C \cap (-C) = \{0\}$, then condition (N) is satisfied: Since $f(0)$ is finite, $A\lambda = 0$ and $\lambda \geq 0$ imply $c\lambda \geq 0$; if $c\lambda = 0$ and $\lambda_i \neq 0$, then $(A_i, c_i) \in C \cap (-C)$, and hence A_i and c_i are 0.]

Let now μ be an extremal marking on the polyhedral cone C such that $\mu(C) = (0, 1)$. Any extreme ray of C is generated by $(0, 1)$ or by an element (A_i, c_i) with $f(A_i) = c_i$. If I denotes the set of all $i \in \{1, \dots, n\}$ such that (A_i, c_i) generates an extreme ray of C , then $A_\mu = \{(A_i, c_i); i \in I\} \cup \{(0, 1)\}$. By Theorem 4.3 there exists an extremal triangulation Σ' of C with respect to μ . We show that the collection $\Sigma = \{p(S); S \in \Sigma'\}$ is a simplicial subdivision of X satisfying (i), whereby p denotes the projection $p: R^{m+1} \rightarrow R^m$, $(x, r) \rightarrow x$. If $S \in \Sigma'$, then S is a convex subcone of C generated by a linearly independent set consisting of $(0, 1)$ and m elements (A_i, c_i) , $i \in I$. Therefore $p(S)$ is a convex subcone of X generated by m linearly independent columns A_i , $i \in I$. From this it follows that $p(S)$ is pointed, too. Because X is the union of Σ , it remains to show that $p(S_1) \cap p(S_2)$ is a face of $p(S_1)$ for $S_1, S_2 \in \Sigma'$. Since every $S \in \Sigma'$ contains $(0, 1)$, this follows easily from the corresponding property of the simplicial subdivision Σ' of C .

(2) The solution function λ and its properties can be derived as follows from the coordinate function $k: C \rightarrow R_+^{A_\mu}$ on C defined by μ (Definition 3.8). For $x \in X$ and $i \in I$ let

$$\lambda_i(x) = k_a((x, f(x))), \quad \text{whereby } a = (A_i, c_i).$$

From (3.15) then we get, because $k_{(0,1)}((x, f(x))) = 0$,

$$(x, f(x)) = \sum_{i \in I} \lambda_i(x) \cdot (A_i, c_i).$$

Hence for $x \in X$ it follows that

$$x = \sum_{i \in I} \lambda_i(x) A_i, \quad f(x) = \sum_{i \in I} \lambda_i(x) c_i,$$

which proves (ii) with $\lambda(x) = (\lambda_i(x))_{i \in I}$. Furthermore $\lambda_j(A_i) = \delta_{ij}$ for $i, j \in I$ because (A_i, c_i) generates an extreme ray of C for $i \in I$.

To prove (iii) we only have to show that λ is additive and positively homogeneous on each cone $p(S)$ of Σ . Then λ is continuous on $p(S)$ and hence continuous on X , since Σ is finite. Note that for any $x \in X$, $r \geq f(x)$,

we have $\lambda_i(x) = k_a((x, r))$ for $a = (A_i, c_i)$. Now let $x, y \in p(S)$, $\alpha \in R_+$, and $z = x + \alpha y$. There exist $r, s \in R$ such that $(x, r), (y, s) \in S$, and by Corollary 3.9

$$\lambda_i(z) = k_a((z, r + \alpha s)) = k_a((x, r)) + \alpha k_a((y, s)) = \lambda_i(x) + \alpha \lambda_i(y).$$

Hence λ is additive and positively homogeneous on $p(S)$. ■

Theorem 4.5 states that the solution of a linear program (P) [satisfying condition (N)] can be made a continuous, positively homogeneous, and piecewise additive function of its right hand sides. For one dimensional parameter variations results of this type were proved fairly early in the development of linear programming [3]. For multidimensional parameter variations the situation is more involved. The above result was first derived by Walkup and Wets [13] by perturbing the objective function in such a way that the perturbed linear program admits unique optimal solutions for all admissible right hand sides. Our approach based on the method of extraction permits a direct treatment of the original problem. However, it requires assumption (N), while the perturbation approach can treat arbitrary linear programs. For a detailed study of multidimensional parameter variations see Nožička et al., [9].

The authors would like to thank a referee for drawing their attention to the paper of V. Klee [8]. In this paper Klee's theorem (see the comments after Lemma 4.1) has been proven for convex sets without completeness of the scalar field by a method which can be viewed as an extraction process without marking.

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